Kei-ichi Maeda[1,2] and David Wands[3]
[1] *Department of Physics, Waseda University, Okubo 3-4-1, Shinjuku, Tokyo 169-8555, Japan*
[2] *Advanced Research Institute for Science and Engineering, Waseda University, Shinjuku, Tokyo 169-8555, Japan*
[3] *Relativity and Cosmology Group, School of Computer Science and Mathematics, University of Portsmouth, Portsmouth PO1 2EG, United Kingdom*

We derive the four-dimensional field equations for the induced metric and scalar field on the world-volume of a 3-brane in a five-dimensional bulk with Einstein gravity plus a self-interacting scalar field. We calculate the effective four-dimensional gravitational constant and cosmological constant for arbitrary forms of the brane tension and self-interaction potential for the scalar field in the bulk. In addition to the canonical energy-momentum tensor for the scalar field and ordinary matter on the brane, the effective four-dimensional Einstein equations include terms due to the scalar field and gravitational waves in the bulk. We present solutions corresponding to static Minkowski brane worlds and also dynamical Friedmann-Robertson-Walker brane world cosmologies. We discuss the induced coupling of the scalar field to ordinary matter on the brane.

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I. INTRODUCTION

There has been tremendous interest over the last year in schemes for dimensional reduction where ordinary matter fields are confined to a lower-dimensional hypersurface, while only gravitational fields propagate throughout all of spacetime. Such speculation has been inspired by recent developments in string/M-theory where gauge-fields can be associated with the degrees of freedom of open strings whose end-points live on a D-brane hypersurface, while gravitational fields associated with closed string excitations, can propagate in the bulk [1–3]. In particular Randall and Sundrum [4] proposed a static model where four-dimensional Newtonian gravity is recovered at low energies when considering linear perturbations about a Minkowski brane embedded in five-dimensional anti-de Sitter spacetime (AdS_5).

Randall and Sundrum [4] considered pure 5-D Einstein gravity in the bulk, sourced only by a cosmological constant, whereas in a string theory or M-theory context one would also expect scalar fields, associated with the many moduli fields, in the gravitational sector, which should, in principle, also be allowed to propagate in the bulk [5–7]. For example, Lukas, Ovrut and Waldram [5] have derived an effective five-dimensional action from a dimensional reduction from 11-dimensional M-theory which contains scalar fields in the bulk which correspond to moduli associated with the compactification of six-dimensions on a Calabi-Yau space.

A particularly elegant way to study the effective four-dimensional gravity induced in so-called 'brane world' models was proposed by Shiromizu, Maeda and Sasaki [8]. They derived the effective 'Einstein equations' for the induced 4-D metric obtained by projecting the five-dimensional metric onto the brane world-volume. This approach yields the most general form of the four-dimensional gravitational field equations for a brane world observer whatever the form of the bulk metric, in contrast to the usual Kaluza-Klein style dimensional reduction which relies on taking a particular form for the bulk metric in order to integrate over the fifth dimension. The price to be paid for such generality, is that the brane world observer may be subject to influences from the bulk, in particular gravitational waves, which are not constrained by local quantities, i.e., the set of four-dimensional equations does not in general form a closed system. Nonetheless, when the brane is located at an orbifold fixed point under Z_2 symmetry (a "brane at the end of the world") the energy-momentum tensor on the brane is sufficient to determine the extrinsic curvature of the brane, and together with the local induced metric, this strongly constrains the brane world gravity. In the absence of bulk matter, the effect of the bulk gravity can be described by the projected five-dimensional Weyl tensor. In many circumstances (e.g., AdS_5 bulk) this can be consistently set to zero.

In this paper we will extend the work of Shiromizu, Maeda and Sasaki [8] to consider a simple dilaton-gravity theory with scalar and tensor fields which propagate in the bulk, with self-interaction potentials for the scalar field in the bulk and on the brane. We will give the general form for the effective scalar field and Einstein equations on the brane, in particular giving expressions for the effective gravitational Newton's constant, and the cosmological constant on the brane, both of which become functions of the local scalar field. We also show how to recover previously found solutions for static Minkowski brane worlds and dynamical Friedmann-Robertson-Walker (FRW) cosmologies.

II. THE FOUR-DIMENSIONAL EFFECTIVE EQUATIONS

Suppose that the i-th brane world is located in the five-dimensional spacetime as

$$Y_{(i)}(X^A) = 0, \tag{2.1}$$

where X^A ($A = 0, 1, 2, 3, 5$) are the 5-dimensional coordinates. The effective action in 5-dimensional spacetime is then

$$S = \int d^5X \sqrt{-{}^{(5)}g} \left[\frac{1}{2\kappa_5^2} {}^{(5)}R - \frac{1}{2}(\nabla\Phi)^2 - {}^{(5)}\Lambda(\Phi) \right] + \sum_i \int_{Y_{(i)}=0} d^4x_{(i)} \sqrt{-g_{(i)}} \left[L_{(i)}^{\rm matter} - \lambda_{(i)}(\Phi) \right], \tag{2.2}$$

where $x^\mu_{(i)}$ ($\mu = 0, 1, 2, 3$) are the induced 4-dimensional brane world coordinates on the i-th brane.

The five-dimensional Einstein equations in the bulk are

$${}^{(5)}G_{AB} = \kappa_5^2 {}^{(5)}T_{AB} \equiv \kappa_5^2 \left[\nabla_A\Phi\nabla_B\Phi - {}^{(5)}g_{AB} \left(\frac{1}{2}(\nabla\Phi)^2 + {}^{(5)}\Lambda \right) \right] \tag{2.3}$$

and the equation of motion for a dilaton Φ is

$${}^{(5)}\Box\Phi - \frac{d^{(5)}\Lambda}{d\Phi} - \sum_i \frac{d\lambda_{(i)}}{d\Phi}\delta(Y_i) = \sum_i \frac{1}{\Omega_{(i)}} \frac{d\Omega_{(i)}}{d\Phi} \tau_{(i)}\, \delta(Y_i)\,, \tag{2.4}$$

where $\tau_{(i)}$ is the trace of the energy-momentum tensor for matter on the i-th brane. This term on the right-hand-side of the equation arises because, in a dilaton-gravity theory, one should allow for the possibility that matter is minimally coupled, not with respect to the original Einstein metric, but with respect to a conformally related metric

$${}^{(5)}\tilde{g}_{AB} = \Omega_{(i)}^2(\Phi)\, {}^{(5)}g_{AB}\,. \tag{2.5}$$

To obtain the basic equations in one of the brane worlds, we project the variables onto the four-dimensional brane world as in Ref. [8]. We shall assume that our four-dimensional world ($i = 1$) is described by a domain wall (3-brane) $(M, g_{\mu\nu})$ at $Y_1 = 0$, which as an orbifold fixed point under Z_2 reflection symmetry, in a five-dimensional spacetime $(\mathcal{M}, {}^{(5)}g_{AB})$ and henceforth drop the index, i. The induced 4D metric is obtained by projecting the five-dimensional metric onto the brane world-volume:

$$g_{AB} = {}^{(5)}g_{AB} - n_A n_B\,, \tag{2.6}$$

where n_A is the spacelike unit-vector field normal to the brane. Using the Gauss and Codazzi equations, we obtain the induced four-dimensional Einstein equations as [8]

$${}^{(4)}G_{\mu\nu} = \frac{2\kappa_5^2}{3} \left[{}^{(5)}T_{AB} g_\mu{}^A g_\nu{}^B + \left({}^{(5)}T_{AB}n^A n^B - \frac{1}{4} {}^{(5)}T^C_C \right) g_{\mu\nu} \right] + KK_{\mu\nu} - K_\mu{}^\sigma K_{\nu\sigma} - \frac{1}{2} g_{\mu\nu} \left(K^2 - K^{\alpha\beta}K_{\alpha\beta} \right) - E_{\mu\nu}, \tag{2.7}$$

where the extrinsic curvature of M is denoted by $K_{\mu\nu} = g_\mu{}^C g_\nu{}^D \nabla_C n_D$, and $K = K^\mu_\mu$ is its trace. The effect of the bulk Weyl tensor is felt on the brane through the symmetric, trace-free tensor [8]

$$E_{\mu\nu} \equiv {}^{(5)}C^E_{AFB} n_E n^F g_\mu{}^A g_\nu{}^B\,. \tag{2.8}$$

The Codazzi equation yields an additional equation

$$D_\nu K_\mu^{\ \nu} - D_\mu K = \kappa_5^2\, {}^{(5)}T_{BC} n^C g_\mu{}^B, \tag{2.9}$$

which turns out to be conservation of energy-momentum on the brane, as we will see later, where D_μ is the covariant derivative with respect to the induced metric $g_{\mu\nu}$.

The equation for ϕ can be reduced to

$$\Box\Phi - a_C\nabla^C\Phi + K\pounds_n\Phi + \pounds_n{}^2\Phi - \frac{d^{(5)}\Lambda}{d\Phi} - \frac{d\lambda}{d\Phi}\delta(Y_1) = \frac{1}{\Omega}\frac{d\Omega}{d\Phi}\tau\delta(Y_1), \tag{2.10}$$

where $\Box \equiv D_\mu D^\mu, a^\mu \equiv n^\nu \nabla_\nu n^\mu$, $\pounds_n$ is the Lie derivative in n direction, and $\lambda \equiv \lambda_{(1)}$.

We can choose a coordinate χ such that the hypersurface $\chi = 0$ coincides with our brane world ($Y_{(1)} = 0$) and $n_A dX^A = d\chi$, which is a condition on the coordinate in the direction of the extra dimension. We assume this choice is possible at least in the neighborhood of the brane. This implies that

$$a^A = 0, \quad \pounds_n \Phi = \Phi', \quad \text{and} \quad \pounds_n^2 \Phi = \Phi'', \tag{2.11}$$

where a prime denotes a differentiation with respect to χ. The assumption of Z_2 symmetry about the brane allows us to expand the dilaton field Φ near the brane as

$$\Phi = \phi(x) + \Phi_1(x)|\chi| + \frac{1}{2}\Phi_2(x)\chi^2 + \mathcal{O}(\chi^3), \tag{2.12}$$

and, inserting this into Eq. (2.10), we find the jump condition

$$\Phi_1 = \frac{1}{2}\left(\frac{d\lambda}{d\phi} + \frac{1}{\Omega}\frac{d\Omega}{d\phi}\tau\right), \tag{2.13}$$

The equation for the dilaton, ϕ, on the brane then becomes

$$\Box\phi + \frac{K}{2}\left(\frac{d\lambda}{d\phi} + \frac{1}{\Omega}\frac{d\Omega}{d\phi}\tau\right) - \frac{d^{(5)}\Lambda}{d\phi} = -\Phi_2\,. \tag{2.14}$$

The junction condition for the extrinsic curvature due to the energy-momentum tensor on the brane, $S_{\mu\nu}$, requires [9]

$$[K_{\mu\nu}] = -{\kappa_5}^2\left(S_{\mu\nu} - \frac{1}{3}g_{\mu\nu}S\right)\,. \tag{2.15}$$

With the ansatz of Z_2 symmetry, we have $[K_{\mu\nu}] = K^+_{\mu\nu} - K^-_{\mu\nu} = 2K_{\mu\nu}$ and hence we obtain

$$K_{\mu\nu} = -\frac{{\kappa_5}^2}{2}\left(S_{\mu\nu} - \frac{1}{3}g_{\mu\nu}S\right). \tag{2.16}$$

Since λ is a four-dimensional vacuum energy, we can split $S_{\mu\nu}$ as $S_{\mu\nu} = -\lambda g_{\mu\nu} + \tau_{\mu\nu}$, where $\tau_{\mu\nu}$ is the energy-momentum tensor derived from the matter Lagrangian on the brane, and we then have

$$\begin{aligned} K_{\mu\nu} &= -\frac{1}{2}{\kappa_5}^2\left(\frac{1}{3}\lambda g_{\mu\nu} + \tau_{\mu\nu} - \frac{1}{3}g_{\mu\nu}\tau\right), \\ K &= -\frac{1}{6}{\kappa_5}^2\left(4\lambda - \tau\right). \end{aligned} \tag{2.17}$$

Finally then, the 4-dimensional effective Einstein equations (2.7) can be rewritten using the bulk energy-momentum tensor, ${}^{(5)}T_{AB}(\Phi)$ given in Eq. (2.3), the jump condition for the scalar field derivative normal to the brane, Eq. (2.13), and the expression for the brane extrinsic curvature in Eq. (2.17), as

$${}^{(4)}G_{\mu\nu} = \frac{2{\kappa_5}^2}{3}\hat{T}_{\mu\nu}(\phi) + \left[-{}^{(4)}\Lambda + \frac{\kappa_5^2}{16}\left(2\frac{d\lambda}{d\phi} + \frac{1}{\Omega}\frac{d\Omega}{d\phi}\tau\right)\frac{1}{\Omega}\frac{d\Omega}{d\phi}\tau\right]g_{\mu\nu} + 8\pi G_N(\phi)\tau_{\mu\nu} + \kappa_5^4\,\pi_{\mu\nu} - E_{\mu\nu}\,, \tag{2.18}$$

where

$$\hat{T}_{\mu\nu} = D_\mu\phi D_\nu\phi - \frac{5}{8}g_{\mu\nu}(D\phi)^2\,, \tag{2.19}$$

$${}^{(4)}\Lambda = \frac{1}{2}\kappa_5^2\left[{}^{(5)}\Lambda + \frac{1}{6}\kappa_5^2\,\lambda^2 - \frac{1}{8}\left(\frac{d\lambda}{d\phi}\right)^2\right], \tag{2.20}$$

$$8\pi G_N = \frac{\kappa_5^4}{6}\lambda(\phi)\,, \tag{2.21}$$

$$\pi_{\mu\nu} = -\frac{1}{4}\tau_{\mu\alpha}{\tau_\nu}^\alpha + \frac{1}{12}\tau\tau_{\mu\nu} + \frac{1}{8}g_{\mu\nu}\tau_{\alpha\beta}\tau^{\alpha\beta} - \frac{1}{24}g_{\mu\nu}\tau^2\,. \tag{2.22}$$

This, of course, has many similarities with the effective Einstein equations on the brane found previously for pure Einstein gravity in the 5-D bulk [8]. The definition of $\pi_{\mu\nu}$, which represents the contribution of terms quadratic in the 4-D energy-momentum tensor, has exactly the same form and when λ and Ω are independent of the dilaton field, ϕ, then the expressions for ${}^{(4)}\Lambda$ and G_N reduce to those found previously [8]. More generally though, the effective Newton's constant on the brane, G_N, becomes a function of the scalar field through the dependence of the brane tension, λ, upon ϕ. Although λ appears to act like an effective Brans-Dicke field, this is not easily related to a four-dimensional Brans-Dicke type gravity theory (e.g., by a conformal transformation) due to scalar field kinetic terms and other non-standard couplings to ϕ that appear in the effective Einstein equations. For instance the trace of the energy-momentum tensor for matter appears at linear order in the Einstein equations when $d\Omega/d\phi \neq 0$, which was recently noted for particular cosmological solutions by Barger et al [10].

Inserting the expression for the trace of the brane extrinsic curvature given in Eq. (2.17) into the equation of motion for the dilaton (2.14) yields

$$\Box\phi = \frac{d}{d\phi}\left({}^{(5)}\Lambda + \frac{\kappa_5^2}{6}\lambda^2\right) - \frac{\kappa_5^2}{12}\left[\frac{d\lambda}{d\phi} - (4\lambda - \tau)\frac{1}{\Omega}\frac{d\Omega}{d\phi}\right]\tau - \Phi_2 \,. \tag{2.23}$$

The function Φ_2 represents the effect of the evolution of the scalar field in the bulk, upon the propagation of the field on the brane. It cannot be determined locally by the field on the brane, without solving the equations of motion in the bulk. This represents a limitation in the uses of the induced dilaton-gravity equations. In the case of vacuum Einstein gravity in the bulk the effect of bulk gravitational field is described solely by the symmetric, trace-free tensor $E_{\mu\nu}$. Symmetry requirements, together with the conservation of energy-momentum on the brane, may be sufficient to completely determine the effect of $E_{\mu\nu}$ in some situations (in particular, homogeneous cosmologies [11,12,8]) without knowing anything about the solution for Einstein equations in the bulk. In the case of scalar-tensor gravity in the bulk, Φ_2 appears as a free-function and will allow energy-momentum conservation for the scalar field on the brane to be violated by transferring energy and/or momentum to the bulk.

Using the bulk energy-momentum tensor, ${}^{(5)}T_{AB}(\Phi)$ given in Eq. (2.3), and the expression for the brane extrinsic curvature in Eq. (2.17) in the Codazzi equation (2.9), we find

$$\frac{1}{2}{\kappa_5}^2\left(\frac{d\lambda}{d\phi}D_\mu\phi - D_\nu{\tau_\mu}^\nu\right) = {\kappa_5}^2 D_\mu\phi\cdot\Phi_1 \,. \tag{2.24}$$

Using the jump condition for the scalar field, given in Eq. (2.13), we obtain

$$D_\nu{\tau_\mu}^\nu = -\frac{1}{\Omega}\frac{d\Omega}{d\phi}\tau D_\mu\phi,$$

which is consistent with conservation of energy-momentum for matter on the brane, $\tilde{D}_\nu\tilde{\tau}^\nu_\mu = 0$, with respect to the conformally related metric $\tilde{g}_{\mu\nu}$ given by Eq. (2.5), in which the conformally rescaled energy-momentum tensor is $\tilde{\tau}_{\mu\nu} = \tau_{\mu\nu}/\Omega^2$.

III. THE DILATON-VACUUM BRANE WORLD

Henceforth we shall consider only dilaton-vacuum solutions on the brane ($\tau_{\mu\nu} = 0$) in which case the effective Einstein equations simplify considerably to give

$${}^{(4)}G_{\mu\nu} = \frac{2{\kappa_5}^2}{3}\hat{T}_{\mu\nu}(\phi) - {}^{(4)}\Lambda g_{\mu\nu} - E_{\mu\nu} \,. \tag{3.1}$$

If we define a quantity $\Delta\Phi_2$, such that

$$\Delta\Phi_2 \equiv \Phi_2 - \frac{1}{4}\frac{d\lambda}{d\phi}\frac{d^2\lambda}{d\phi^2} \,, \tag{3.2}$$

then we can rewrite the equation of motion of the dilaton on the brane (2.23) as

$$\Box\phi - \frac{dU_{eff}}{d\phi} = -\Delta\Phi_2 \,, \tag{3.3}$$

where the effective potential for the scalar field on the brane is

$$U_{eff} = \frac{2}{\kappa_5^2} {}^{(4)}\Lambda_4 = {}^{(5)}\Lambda + \frac{1}{6}\kappa_5^2\,\lambda^2 - \frac{1}{8}\left(\frac{d\lambda}{d\phi}\right)^2 \tag{3.4}$$

Introducing the canonical form of the energy-momentum tensor for the scalar field on the brane as

$$T_{\mu\nu}(\phi) = D_\mu\phi D_\nu\phi - g_{\mu\nu}\left(\frac{1}{2}(D\phi)^2 + U_{eff}\right), \tag{3.5}$$

and a vector field $J_\mu = \Delta\Phi_2 \cdot D_\mu\phi$, we find that Eq.(3.3) yields

$$D^\nu T_{\mu\nu} = -J_\mu\,. \tag{3.6}$$

Thus we regard J_μ as the energy-momentum lost from the scalar field on the brane to the bulk.

The four-dimensional Einstein equations Eq.(3.1) are now

$${}^{(4)}G_{\mu\nu} = \frac{2{\kappa_5}^2}{3}\left[T_{\mu\nu}(\phi) + \Delta T_{\mu\nu}(\phi)\right] - E_{\mu\nu}\,, \tag{3.7}$$

where the contribution from the scalar field energy density in the bulk is given by

$$\Delta T_{\mu\nu} = \frac{1}{4} g_{\mu\nu}\left(U_{eff} - \frac{1}{2}(D\phi)^2\right). \tag{3.8}$$

This has the same equation of state as a cosmological constant, albeit with a time-dependent value proportional to the Hamiltonian density for the scalar field on the brane.

From the Einstein equations (3.7) and the Bianchi identity, we have

$$\begin{aligned} D^\nu E_{\mu\nu} &= \frac{2{\kappa_5}^2}{3} D^\nu\left[T_{\mu\nu}(\phi) + \Delta T_{\mu\nu}(\phi)\right] \\ &= \frac{2{\kappa_5}^2}{3}\left[D^\nu\left(\Delta T_{\mu\nu}\right) - J_\mu\right]. \end{aligned} \tag{3.9}$$

A. Minkowski brane worlds

We shall now demonstrate that we can recover known static solutions with four-dimensional Minkowski spacetime on the brane in the absence of matter ($\tau_{\mu\nu} = 0$) when there is no energy transfer between bulk and brane ($J_\mu = 0$) for particular forms of the bulk vacuum energy, ${}^{(5)}\Lambda$, and brane tension, λ.

The Randall-Sundrum brane is a slice in five-dimensional anti-de Sitter spacetime [13,4,14] with constant dilaton field (both in the bulk, $\Phi_1 = 0$, $\Phi_2 = 0$, and on the brane, $D_\mu\phi = 0$) and vanishing Weyl curvature in the bulk, and hence $E_{\mu\nu} = 0$ on the brane. We then recover four-dimensional Minkowski spacetime on the brane [with ${}^{(4)}G_{\mu\nu} = 0$ in Eq. (2.18) and $\Box\phi = 0$ in Eq. (3.3)] when the vacuum energy in the bulk and brane tension are both independent of Φ and obey the Randall-Sundrum condition [13,4]

$${}^{(4)}\Lambda = \frac{1}{2}\kappa_5^2\left[{}^{(5)}\Lambda + \frac{1}{6}\kappa_5^2\,\lambda^2\right] = 0\,. \tag{3.10}$$

Indeed we see that any vacuum solution of four-dimensional general relativity is a particular solution in this case.

In the case of a vanishing bulk cosmological constant, ${}^{(5)}\Lambda = 0$, one can also obtain Minkowski spacetime on the brane with vanishing effective cosmological constant on the brane, ${}^{(4)}\Lambda = 0$, even when the brane tension is non-zero. From Eq. (2.20) we require

$$\frac{\kappa_5^2}{6}\lambda^2 = \frac{1}{8}\left(\frac{d\lambda}{d\phi}\right)^2 \tag{3.11}$$

and hence $\lambda \propto \exp(\pm 2\kappa_5\phi/\sqrt{3})$. For a coupling with this specific functional dependence upon the bulk scalar, the vanishing of the four-dimensional cosmological constant is independent of the actual amplitude of the brane tension. In particular it may be independent of radiative corrections to the brane tension due to matter fields on the brane offering a possible resolution of the cosmological constant problem [15,16]. However the vanishing of ${}^{(4)}\Lambda$ is sensitive to corrections to the vacuum energy in the bulk [17].

In the case of a non-vanishing bulk potential the existence of a Minkowski brane requires

$$
{}^{(5)}\Lambda + \frac{\kappa_5^2}{6}\lambda^2 = \frac{1}{8}\left(\frac{d\lambda}{d\phi}\right)^2 \tag{3.12}
$$

An example is provided by the five-dimensional effective action obtained by Lukas, Ovrut and Waldram [5] from a dimensional reduction of an eleven-dimensional Hořava-Witten model, in which ${}^{(5)}\Lambda = (\alpha_0^2/6{\kappa_5}^2)e^{-2\sqrt{2}\kappa_5\Phi}$ and $\lambda = (\sqrt{2}\alpha_0/{\kappa_5}^2)e^{-\sqrt{2}\kappa_5\Phi}$. Note that in this case the form of the action in Eq. (2.2) is invariant under a global rescaling where $\Phi \to \Phi$+constant and $\alpha_0 \to \alpha_0\times$constant [23]. We will return to this example when we discuss non-static solutions later. For a constant scalar field to to be a solution to the equation of motion for the scalar field on the brane, Eq. (3.3), we require in addition that $J_\mu = 0$ and hence, from Eq. (3.2),

$$
\Phi_2 = \frac{1}{4}\frac{d\lambda}{d\phi}\frac{d^2\lambda}{d\phi^2}\,. \tag{3.13}
$$

As Φ_2 appears as a free function on the brane this is not a restriction on the theory parameters, but one must solve the full five-dimensional equations of motion to determine whether it is a consistent solution in the bulk [5].

Quite generally, if one seeks a static solution in the bulk of the form

$$
ds_5^2 = e^{2A(\chi)}\eta_{\mu\nu}dx^\mu dx^\nu + d\chi^2\,, \tag{3.14}
$$

where $e^{2A(y)}$ is commonly called the "warp factor", then the five-dimensional Einstein equations (2.3) admit a solution of the form [18,19,10]

$$
A' = -\frac{1}{6}W(\Phi)\,, \qquad \Phi' = \frac{1}{2\kappa_5^2}\frac{dW}{d\Phi}\,. \tag{3.15}
$$

where the auxiliary field (or "superpotential") $W(\Phi)$ is related to the bulk potential via the relation

$$
\frac{1}{4\kappa_5^2}\left(\frac{dW}{d\Phi}\right)^2 - \frac{1}{3}W^2 = 2\kappa_5^{2\,(5)}\Lambda \tag{3.16}
$$

The jump conditions at the brane, given in Eqs. (2.17) and (2.13) require in addition that

$$
W(\phi) = \kappa_5^2\lambda(\phi)\,. \tag{3.17}
$$

For a static brane solution with $\phi = \phi_c$ fixed then this requires only that $W(\phi_c) = \kappa_5^2\lambda(\phi_c)$ at that particular value. But if this relation holds for all field values, ϕ, then inserting this relation into Eq. (3.16) then shows that Eq. (3.12) for vanishing cosmological constant on the brane is satisfied on any χ =constant hypersurface. On the other hand substituting this relation into Eq. (3.15) yields the required form for Φ_2 given in Eq. (3.13).

B. FRW dilaton-vacuum brane cosmologies

It is also possible to find Friedmann-Robertson-Walker (FRW) cosmological solutions on the brane with time-dependent scale factor, $a(t)$, and scalar field, $\phi(t)$, where t is cosmic proper time, if we know, or make some assumption about the energy transfer from brane to bulk, $J_0 = \dot\phi\,\Delta\Phi_2$.

The scalar field equation of motion (3.3) is

$$
\ddot\phi + 3H\dot\phi = \frac{dU_{eff}}{d\phi} + \Delta\Phi_2\,, \tag{3.18}
$$

where $H = \dot a/a$ is the Hubble expansion parameter, while the Friedmann equation is

$$
H^2 + \frac{k}{a^2} = \frac{2{\kappa_5}^2}{9}\left(\rho_\phi + \Delta\rho_\phi\right) + \frac{1}{3}E^0{}_0\,. \tag{3.19}
$$

The energy density of the scalar field ρ_ϕ and the effect from the scalar field in the bulk $\Delta\rho_\phi$ are given by

$$\rho_\phi = -T^0_{\ 0} = \frac{1}{2}\dot{\phi}^2 + U_{eff} \quad \text{and} \quad \Delta\rho_\phi = -\Delta T^0_{\ 0} = -\frac{1}{4}\rho_\phi \tag{3.20}$$

The equation for $E^0_{\ 0}$ is given from Eq. (3.9) as

$$\dot{E^0_{\ 0}} + 4HE^0_{\ 0} = \frac{2{\kappa_5}^2}{3}\left[\frac{1}{4}\dot{\rho}_\phi - J_0\right], \tag{3.21}$$

where we have used $E^\mu_{\ \mu} = 0$ and $\Delta T_{\mu\nu} = (\rho_\phi/4)g_{\mu\nu}$.

1. Solutions with energy conservation for dilaton field on the brane

When there is no energy transfer from brane to bulk, i.e. $J_0 = 0$, we find a closed set of equations (3.18–3.21) for the dilaton-vacuum universe, for a given U_{eff}. Equations (3.18) and (3.21) can then be integrated, if the bulk and brane potentials obey the generalized Randall-Sundrum condition given in Eq. (3.12), so that $U_{eff} = 0$. Equation (3.18) can be simply integrated to give

$$\dot{\phi} = \frac{C_\phi}{a^3} \tag{3.22}$$

where C_ϕ is an integration constant, and Eq. (3.21) becomes

$$\dot{E^0_{\ 0}} + 4HE^0_{\ 0} = \frac{2{\kappa_5}^2}{3}\frac{d}{dt}\left(\frac{C_\phi^2}{8a^6}\right). \tag{3.23}$$

This in turn can be integrated to give

$$E^0_{\ 0} = \frac{{\kappa_5}^2 C_\phi^2}{4a^6} + \frac{\mathcal{E}_0}{a^4}, \tag{3.24}$$

where $\mathcal{E}_0$ is another integration constant. Inserting Eqs. (3.22) and (3.24) into Eq. (3.19), we find

$$H^2 + \frac{k}{a^2} = \frac{{\kappa_5}^2 C_\phi^2}{6a^6} + \frac{\mathcal{E}_0}{3a^4}. \tag{3.25}$$

This is the same as the standard Friedmann equation with stiff matter, with energy density proportional to a^{-6}, and radiation, with energy density is proportional to a^{-4}.

The general solution with an initial singularity ($a = 0$) can be expressed in terms of the conformal time, $\eta = \int dt/a$, as [20]

$$a^2 = \frac{\tau\left(\sqrt{6}\kappa_5|C_\phi| + \mathcal{E}_0\tau\right)}{3\left(1 + k\tau^2\right)}, \tag{3.26}$$

where

$$\tau \equiv \begin{cases} |\eta| & \text{for } k = 0 \\ \tan|\eta| & \text{for } k = +1 \\ \tanh|\eta| & \text{for } k = -1 \end{cases}, \tag{3.27}$$

and

$$\sqrt{\frac{2}{3}}\kappa_5(\phi - \phi_0) = \pm\ln\left|\frac{\mathcal{E}_0\tau}{\sqrt{6}\kappa_5|C_\phi| + \mathcal{E}_0\tau}\right|, \tag{3.28}$$

for $\mathcal{E}_0 \neq 0$, or

$$\sqrt{\frac{2}{3}}\kappa_5(\phi - \phi_0) = \pm\ln|\tau|, \tag{3.29}$$

TABLE I. The initial and final asymptotic behaviors of the universe with $C_\phi \neq 0$ for each parameter range of k and $\mathcal{E}_0$. There are two solutions for the case of $k = -1$ and $\mathcal{E}_0 \leq -\sqrt{6}\kappa_5|C_\phi|$. [BC] means a big crunch, which occurs at $t = t_c$. The universe starts to expand from a big bang singularity and eventually collapses to a big crunch singularity after reaching a maximum radius. [SF] means a singularity-free solution which has no initial or final singularity. Such a universe may initially collapse from an infinite radius at $t = -\infty$, and bounce at a finite radius a_0, and then expand to an infinite radius as $t \to \infty$. Note that $a = a_0$= constant if $\mathcal{E}_0 = -\sqrt{6}\kappa_5|C_\phi|$.

	$\mathcal{E}_0 > 0$	$\mathcal{E}_0 = 0$	$\mathcal{E}_0 < 0$	
			$-\sqrt{6}\kappa_5\lvert C_\phi\rvert < \mathcal{E}_0 < 0$	$\mathcal{E}_0 \leq -\sqrt{6}\kappa_5\lvert C_\phi\rvert$
$k = -1$	$t^{1/3} \Rightarrow t$	$t^{1/3} \Rightarrow t$	$t^{1/3} \Rightarrow t$	$t^{1/3} \Rightarrow (t_c - t)^{1/3}$ [BC] $a_0(\neq 0) \Rightarrow t$ [SF]
$k = 0$	$t^{1/3} \Rightarrow t^{1/2}$	$t^{1/3}$	$t^{1/3} \Rightarrow (t_c - t)^{1/3}$ [BC]	
$k = 1$	$t^{1/3} \Rightarrow (t_c - t)^{1/3}$ [BC]	$t^{1/3} \Rightarrow (t_c - t)^{1/3}$ [BC]	$t^{1/3} \Rightarrow (t_c - t)^{1/3}$ [BC]	

when $\mathcal{E}_0 = 0$, where $\pm$ corresponds to the sign of C_ϕ. The initial singularity appears at $\tau = 0$. We also find a big crunch at $\tau = \tau_c \equiv \sqrt{6}\kappa_5|C_\phi|/|\mathcal{E}_0|$ if $\mathcal{E}_0 < 0$ for $k = 0$ and 1 and if $\mathcal{E}_0 \leq -\sqrt{6}\kappa_5|C_\phi|$ for $k = -1$.

Solutions with $C_\phi \neq 0$ start expanding away from the initial singularity with

$$a = \left(\frac{3}{2}{\kappa_5}^2 {C_\phi}^2\right)^{1/6} t^{1/3} \tag{3.30}$$

This is just the standard cosmological solution for a spatially flat universe with stiff matter. Since we have a massless dilaton field, we naturally expect such a solution as a particular solution. Spatially flat models ($k = 0$) with $\mathcal{E}_0 > 0$ then approach the standard radiation-dominated evolution with

$$a = \left(\frac{4\mathcal{E}_0}{3}\right)^{1/4} t^{1/2}, \tag{3.31}$$

but if $\mathcal{E}_0 < 0$ then the universe will recollapse to a big crunch when the conformal time reaches $\eta = \sqrt{6}\kappa_5|C_\phi|/|\mathcal{E}_0| > 0$. Although the similar big crunch appears for $k = -1$ if $\mathcal{E}_0 \leq -\sqrt{6}\kappa_5|C_\phi|$, a non-singular solution also exists in the same parameter range. In particular, if $C_\phi = 0$, the open model with $k = -1$ and $\mathcal{E}_0 < 0$ is always non-singular.

There is another exact solution for $k = -1$ when $\mathcal{E}_0 \leq -\sqrt{6}\kappa_5|C_\phi|$, which is

$$a^2 = \frac{{a_0}^2}{(1-\tau^2)}\left[1 - \left(\frac{\tau}{\tau_*}\right)^2\right], \tag{3.32}$$

and

$$(\phi - \phi_0) = -3C_\phi \ln\left|\frac{\tau_* - \tau}{\tau_* + \tau}\right|, \tag{3.33}$$

where

$$\begin{aligned} \tau &= \tanh|\eta| \\ a_0^2 &= \frac{1}{6}\left[|\mathcal{E}_0| + \sqrt{\mathcal{E}_0^2 - 6{\kappa_5}^2{C_\phi}^2}\right] \\ \tau_* &= \frac{1}{\sqrt{6}\kappa_5|C_\phi|}\left[|\mathcal{E}_0| + \sqrt{\mathcal{E}_0^2 - 6{\kappa_5}^2{C_\phi}^2}\right] \ (\geq 1). \end{aligned} \tag{3.34}$$

This solution is non-singular: that is, the infinitely large universe ($a = \infty$) contracting from the past infinity ($t = -\infty$), bounces at $t = 0$ with a finite scale factor a_0, and then expands to $a = \infty$ as $t \to \infty$.

We summarize the asymptotic behaviors of the solutions in Tables 1 and 2 for the cases of $C_\phi \neq 0$ and $C_\phi = 0$, respectively.

2. Solutions with time-dependent radion

While the assumption of no energy transfer to or from the scalar field on the brane gives simple cosmological solutions on the brane, these differ from previously derived solutions for the full five-dimensional dilaton-gravity found by Lukas et al [5]. They considered a five-dimensional metric

TABLE II. The initial and final asymptotic behaviors of the universe with $C_\phi = 0$ for each parameter range of k and $\mathcal{E}_0$. [BC] means a big crunch at $t = t_c$. [SF] means a singularity-free solution.

	$\mathcal{E}_0 > 0$	$\mathcal{E}_0 = 0$	$\mathcal{E}_0 < 0$
$k = -1$	$t^{1/2} \Rightarrow t$	t [Milne universe]	$a_0(\neq 0) \Rightarrow t$ [SF]
$k = 0$	$t^{1/2}$	a_0 [Minkowski space]	no solution
$k = 1$	$t^{1/2} \Rightarrow (t_c - t)^{1/2}$ [BC]	no solution	no solution

$$ds_5{}^2 = -n^2 dt^2 + a^2 d\mathbf{x}^2 + b^2 dy^2 \tag{3.35}$$

where n, a and b, and the scalar field Φ are all separable functions of t and y. In this case the five dimensional equations can be related to a four-dimensional effective theory by a Kaluza-Klein-type dimensional reduction where one integrates out the y-dependence. Although the radion b is non-minimally coupled to the 4D metric and scalar, ϕ, one can recover standard 4D Einstein gravity with minimally coupled radion, by working in terms of the conformally transformed time $\bar{t}$ and scale factor, $\bar{a}$, where

$$ds_5{}^2 = b^{-1}\left(-n^2 d\bar{t}^2 + \bar{a}^2 d\mathbf{x}^2\right) + b^2 dy^2\,. \tag{3.36}$$

Conversely, working in the original frame, one finds an energy transfer between the scalar field on the brane, $\phi(t)$, and the radion field b, described by $J_0 \propto (\dot{b}/b)\rho_\phi$. [1] At the same time the expansion of the bulk metric is itself determined by the local density. In order to obtain separable solutions for the bulk metric, Lukas et al [5] require $\ln b \propto \Phi$ in the bulk, corresponding to $J_0 \propto \dot{\phi}^3$ on the brane.

If we describe the brane to bulk energy transfer by

$$J_0 = -\sqrt{2}\Gamma\kappa_5\dot{\phi}^3\,, \tag{3.37}$$

where Γ is a constant, then Eq. (3.18) can be integrated to give

$$\dot{\phi} = \frac{C_\phi}{a^3 e^{\sqrt{2}\Gamma\kappa_5\phi}}\,, \tag{3.38}$$

which reduces to our previous solution given in Eq. (3.22) when $\Gamma = 0$. The remaining equations (3.21) and (3.19) can be integrated if we make the power-law ansatz

$$a \propto |t|^p \quad \text{and} \quad e^{\sqrt{2}\kappa_5\phi} \propto |t|^q\,, \tag{3.39}$$

where the dilaton evolution Eq. (3.18) requires $3p + \Gamma q = 1$. Equation (3.21) yields

$$E_0^0 = -\frac{(4p-1)q^2}{8(2p-1)}|t|^{-2} + \frac{\mathcal{E}_0}{|t|^{4p}}. \tag{3.40}$$

The second term in the right hand side is the "dark radiation" term, which is proportional to a^{-4}. For consistency with our power-law ansatz, we require that $\mathcal{E}_0$ must be zero except for $p = 1/2$. Inserting the solution (3.40) into Eq.(3.19) for $p \neq 1/2$, we find $12p(2p-1) = -q^2$. Combined with the requirement that $3p + \Gamma q = 1$ this yields the quadratic

$$3(3 + 8\Gamma^2)p^2 - 6(1 + 2\Gamma^2)p + 1 = 0\,. \tag{3.41}$$

We then find a one-parameter family of power-law solutions (3.39) with exponents (see Figure 1)

$$p = p_{(\pm)} = \frac{3(2\Gamma^2+1) \pm 2\sqrt{3}\Gamma\sqrt{3\Gamma^2+1}}{3(8\Gamma^2+3)} \tag{3.42}$$

$$q = q_{(\pm)} = \frac{2[\Gamma \mp \sqrt{3(3\Gamma^2+1)}]}{8\Gamma^2+3}. \tag{3.43}$$

[1]This ties in with our expectation that the energy density of the scalar field in the bulk, (which contributes to the effective Einstein equations on the brane through the non-local $\Delta T_{\mu\nu}$ and $E_{\mu\nu}$ tensors) will change with the expansion/contraction of the bulk metric.

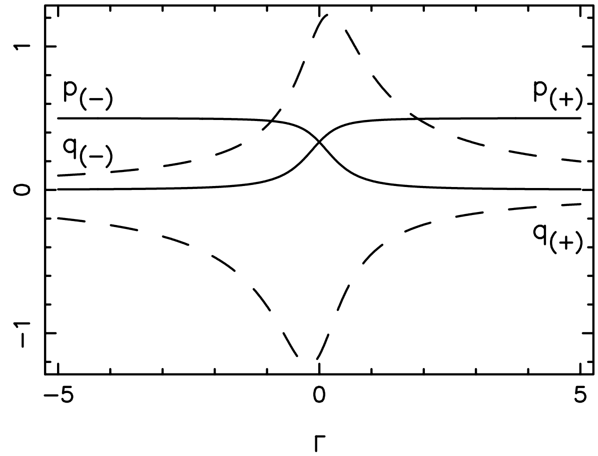

FIG. 1. Plot of the values of the exponents $p_{(\pm)}$ (solid lines) and $q_{(\pm)}$ (dashed lines) given in Eqs. (3.42) and (3.43).

For $\Gamma = 1$ we recover the solutions of Lukas et al [5] with

$$p_{(\pm)} = \frac{3}{11}\left(1 \pm \frac{4\sqrt{3}}{9}\right) \quad \text{and} \quad q_{(\pm)} = \frac{2}{11}\left(1 \mp 2\sqrt{3}\right) . \tag{3.44}$$

These solutions were generalised to spatially curved FRW models including an additional bulk scalar field by Reall [21], and for different potential exponents by Lidsey [22]. In the notation of Ref. [23] they are type II solutions.

The "dynamical instability" of the self-tuning potentials [obeying the condition given in Eq. (3.11)] reported in Ref. [24] is recovered for $\Gamma = \pm\sqrt{1/24}$ where we obtain $p = 1/4$ and $q = \pm\sqrt{3/2}$. The 4D observer sees this as the runaway behaviour of a massless scalar field in a contracting cosmology for $t < 0$ which is always losing energy to the bulk ($J_0 < 0$).

We recover the stiff-matter dominated solution given in Eq. (3.30) with $p_{(\pm)} = 1/3, q_{(\pm)} = \mp 2/\sqrt{3}$ for $\Gamma = 0$. Over the entire range $-\infty < \Gamma < \infty$, $p_{(+)}$ changes monotonically from 0 to 1/2, while $q_{(+)}$ always takes negative values from zero to zero, having its minimum value $-\sqrt{6}/2$ at $\Gamma = -\sqrt{6}/12$. Note that the solution with $(p_{(-)}(\Gamma), q_{(-)}(\Gamma))$ is the same as that obtained by changing the sign of Γ, i.e. the solution with $(p_{(+)}(-\Gamma), -q_{(+)}(-\Gamma))$.

IV. CONCLUSION AND REMARKS

In this paper we have derived the equations of motion for the induced metric and the scalar field on a four-dimensional brane world embedded in five-dimensional Einstein gravity plus a scalar field, Φ, with self-interaction potential energy $\Lambda_5(\Phi)$ and brane tension $\lambda(\Phi)$. In the case where Φ = constant we recover the modified Einstein equations first derived in Ref. [8].

More generally, we find that the induced cosmological constant on the brane is given by the sum of three parts (i) the bulk cosmological constant, Λ_5, (ii) the brane tension, which yields extrinsic curvature of the brane, λ^2, and, (iii) the first derivative of the brane tension, which leads to a discontinuity in the scalar field gradient normal to the brane. The resulting effective cosmological constant is given in Eq. (3.4) as

$$^{(4)}\Lambda_4 = \frac{\kappa_5^2}{2}\left[{}^{(5)}\Lambda + \frac{1}{6}\kappa_5^2\,\lambda^2 - \frac{1}{8}\left(\frac{d\lambda}{d\phi}\right)^2\right] . \tag{4.1}$$

Setting this to zero generalises the Randall-Sundrum condition [13,4] (recovered for $d\lambda/d\phi = 0$), needed to obtain 4D Minkowski space-time solutions on the brane. This is automatically satisfied for all values of the scalar-field in supergravity theories [5,18] where the bulk and brane potentials are derived from the same superpotential. A special case is provided by the self-tuning potentials [15,16] for which $\Lambda_5 = 0$ and $\lambda \propto e^{\pm 2\phi/\sqrt{3}}$. It is interesting to note that a static solution does not require a minimum for either Λ_5 or λ, but only that the induced $^{(4)}\Lambda_4$ has a stationary value.

In addition to the canonically defined energy-momentum tensor for the scalar field on the brane, the energy density of the scalar and gravitational waves in the bulk also contribute to the 4D spacetime curvature. The energy transfer between brane and bulk as seen by the 4D observer is parameterised by the 4-vector $J_\mu = (D_\mu\phi)\Delta\Phi_2$, which cannot be determined by the 4D equations but must be determined from the full 5D solution. Nonetheless we are able to recover known FRW cosmological solutions with time-dependent $\phi(t)$ by parameterising $J_0 \propto \dot{\phi}^3$.

In this paper we have focussed upon dilaton-vacuum solutions on the brane, but in the presence of ordinary matter, the brane tension, $\lambda(\phi)$, determines the strength of the gravitational (Newton's) constant on the brane, $G_N \propto \lambda$. The evolution equation for the scalar field then couples the field on the brane to the trace of the energy-momentum tensor, τ, for ordinary matter on the brane. From Eqs. (2.23) and (3.6) we obtain

$$J_\mu = (D_\mu\phi)\left\{\Delta\Phi_2 + \frac{\kappa_5^2}{12}\left[\frac{d\lambda}{d\phi} - (4\lambda-\tau)\frac{1}{\Omega}\frac{d\Omega}{d\phi}\right]\tau\right\}, \tag{4.2}$$

where we allow for the possibility that matter may be minimally coupled in a conformally related metric $\tilde{g}_{\mu\nu} = \Omega^2(\phi)g_{\mu\nu}$. Such a picture has important consequences for any attempt to understand the present value of the 4D cosmological constant, and its relation to the 4D Planck scale, in a brane world context [15–19,10,24,25].

While this work was being completed we became aware of related work by Barceló and Visser [26] and Mennim and Battye [27].

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